Large-N Limit of $\mathcal{N} = 2$ Supersymmetric Q^N Model in Two Dimensions

Kiyoshi Higashijima^{a*}, **Tetsuji Kimura**^{a†}, **Muneto Nitta**^{b‡} **and Makoto Tsuzuki**^{a§}

^{a}*Department of Physics, Graduate School of Science, Osaka University,*

Toyonaka, Osaka 560-0043, Japan

^{b}*Department of Physics, Tokyo Institute of Technology,*

Oh-okayama, Meguro, Tokyo 152-8551, Japan

Abstract

We investigate non-perturbative structures of the two-dimensional $\mathcal{N} = 2$ supersymmetric non-linear sigma model on the quadric surface $Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$, which is a Hermitian symmetric space, and therefore Kähler, by using the auxiliary field and large-N methods. This model contains two kinds of non-perturbatively stable vacua; one of them is the same vacuum as that of supersymmetric $\mathbf{C}P^{N-1}$ model, and the other is a new kind of vacuum, which has not yet been known to exist in two-dimensional nonlinear sigma models, the Higgs phase. We show that both of these vacua are asymptotically free. Although symmetries are broken in these vacua, there appear no massless Nambu-Goldstone bosons, in agreement with Coleman's theorem, due to the existence of two different mechanisms in these vacua, the Schwinger and the Higgs mechanisms.

* E-mail: higashij@phys.sci.osaka-u.ac.jp

† E-mail: t-kimura@het.phys.sci.osaka-u.ac.jp

‡ E-mail: nitta@th.phys.titech.ac.jp

§ E-mail: tsuzki@het.phys.sci.osaka-u.ac.jp

1 Introduction

Non-perturbative analyses in quantum field theories and string theories have been recognized to be necessary to solve important problems which cannot be solved in the frameworks of perturbative theories. In some cases, degenerate vacua that are found to be stable in perturbative analyses turn out to be false vacua as a result of non-perturbative effects. Recently, there has been much progress in supersymmetric gauge field theories in four dimensions [1].

Two-dimensional nonlinear sigma models have attracted interest because of their similarity to four-dimensional gauge field theories, such as mass gaps, asymptotic freedom, instantons and so on (see, e.g., Refs. [2, 3, 4, 5] for a review). For this reason it is interesting to investigate non-perturbative effects in two-dimensional nonlinear sigma models. If we reformulate nonlinear sigma models by using auxiliary fields, we can investigate non-perturbative effects easily with large-N methods. In the $O(N)$ model, we can find a mass gap, in contrast to perturbative analyses. In the $\mathbf{C}P^{N-1}$ model, there is a mass gap, a gauge boson is dynamically generated, and confinement due to this generated gauge boson occurs [6].

The $\mathcal{N} = 1$ supersymmetric $O(N)$ model consists of the bosonic $O(N)$ model and the Gross-Neveu model. Since the Gross-Neveu model illustrates dynamical chiral symmetry breaking, the $\mathcal{N} = 1$ supersymmetric $O(N)$ model also has this property [7]. In principle, bosonic and $\mathcal{N} = 1$ supersymmetric nonlinear sigma models on an arbitrary coset space G/H can be formulated using auxiliary fields to study non-perturbative effects.

What about $\mathcal{N} = 2$ supersymmetric nonlinear sigma models in two dimensions? These models may possess similarities to four-dimensional $\mathcal{N} = 2$ QCD. However, only a few models have been investigated to this time, since there is no auxiliary field formulation of $\mathcal{N} = 2$ supersymmetric nonlinear sigma models, except for the $\mathbf{C}P^{N-1}$ model and the Grassmannian model [8, 9, 10]. One of difficulties in such an investigation is the fact that target manifolds of $\mathcal{N} = 2$ supersymmetric nonlinear sigma models must be Kähler manifolds [11]. However, two of the present authors have recently given an auxiliary field formulation of four-dimensional $\mathcal{N} = 1$ supersymmetric nonlinear sigma models (which are equivalent to two-dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models through dimensional reduction), whose target spaces are the Hermitian symmetric spaces summarized in Table 1 [12, 13]. (For a review, see Ref. [14].) The Hermitian symmetric spaces, which include the $\mathbf{C}P^{N-1}$ and the Grassmann manifold, are in a familiar class of Kähler coset spaces G/H, whose (nonlinear) Kähler potentials can be constructed by supersymmetric nonlinear realization methods [15]. As mentioned above, non-perturbative effects of the $\mathcal{N} = 2$ supersymmetric $\mathbf{C}P^{N-1}$ model have been studied in

Type	G/H	$\dim_{\mathbf{C}}(G/H)$
AIII_1	$\mathbf{C}P^{N-1} = SU(N)/SU(N-1) \times U(1)$	$N-1$
AIII_2	$G_{N,M}(\mathbf{C}) = U(N)/U(N-M) \times U(M)$	$M(N-M)$
BDI	$Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$	$N-2$
CI	$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)$
DIII	$SO(2N)/U(N)$	$\frac{1}{2}N(N-1)$
EIII	$E_6/SO(10) \times U(1)$	16
EVII	$E_7/E_6 \times U(1)$	27

Table 1: Hermitian symmetric spaces.

The left column gives the classification by Cartan. The first three manifolds, $\mathbf{C}P^{N-1}$, $G_{N,M}(\mathbf{C})$ and $Q^{N-2}(\mathbf{C})$, are called a (complex) projective space, a (complex) Grassmann manifold, and a (complex) quadric surface, respectively.

a large number of works (see Refs. [8, 9] and papers that cite them), since there already exists the auxiliary field formulation of $\mathbf{C}P^{N-1}$. The other models have not been studied yet. The investigation of new models should provide deeper knowledge about non-perturbative effects of quantum field theories. The purpose of this paper is to investigate non-perturbative effects of one of the new models, the quadric surface $Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$, by using auxiliary field methods and large-N methods (the leading order of the $1/N$ expansion). We call this model simply the "Q^N model" in this paper.

We introduce auxiliary vector and chiral superfields to reformulate the nonlinear sigma model. If we integrate out auxiliary superfields, we obtain the original nonlinear sigma model again. The integration over auxiliary vector and chiral superfields gives D-term and F-term constraints, respectively. If we set all of the auxiliary *chiral* superfields to zero, we obtain the $\mathbf{C}P^{N-1}$ model. If we set the auxiliary *vector* superfield to zero, we obtain the the non-compact $\mathcal{N} = 2$ supersymmetric $O(N)$ sigma model, which is a generalization of bosonic and $\mathcal{N} = 1$ supersymmetric $O(N)$ models [16]. By integrating out the dynamical fields, and calculating the effective action and the effective potential, we investigate non-perturbatively stable vacua of the Q^N model. We find that there exist two stable vacua: one is the vacuum known in the $\mathcal{N} = 2$ supersymmetric $\mathbf{C}P^{N-1}$ model, and the other is a new type of vacuum which is found here for the first time.

We find that, in both phases, there is mass gap, and auxiliary superfields become dynamical as bound states of original dynamical fields for large N. In particular, a gauge boson is dynamically generated as in the $\mathbf{C}P^{N-1}$ model. Moreover, we show that both phases are asymptotically free by

calculating the beta function. One of the key points which we should elucidate in both vacua is the disappearance of massless Nambu-Goldstone bosons. In two dimensions, the existence of massless Nambu-Goldstone bosons is forbidden by Coleman's theorem [17]. We can avoid this problem owing to several mechanisms, supersymmetry, the Schwinger mechanism, and the Higgs mechanism.

This paper is organized as follows. In section 2 we formulate the Q^N model with auxiliary superfields using the notation of four-dimensional $\mathcal{N} = 1$ supersymmetry. We perform dimensional reduction to two dimensions and summarize the symmetries of the two-dimensional Lagrangian. We calculate the effective potential and find two kinds of non-perturbatively stable vacua in section 3. In section 4 we discuss one of the stable vacua, which we call the Schwinger phase. In section 5 we investigate the other stable vacuum, which is a new kind of vacuum, the Higgs phase. We devote section 6 to conclusions and discussion. We summarize the notation of $\mathcal{N} = 1$ supersymmetry in four dimensions in Appendix A. Appendix B describes the dimensional reduction to $\mathcal{N} = 2$ supersymmetry in two dimensions.

2 Auxiliary Field Formulation of the Q^N Model

In this section we formulate the Q^N model with auxiliary superfields in four-dimensional $\mathcal{N} = 1$ supersymmetry notation. We then perform the dimensional reduction to two dimensions following the prescription given in Appendix B. We discuss symmetries of this Lagrangian in the second subsection.

2.1 Lagrangian in four dimensions and reduction to two dimensions

First, we give the auxiliary field formulation of the Q^N model in $\mathcal{N} = 1$ four-dimensional notation [12, 13]. Let $\Phi_i(x, \theta, \bar{\theta})$ $(i = 1, \cdots, N)$ be dynamical chiral superfields belonging to the $SO(N)$ vector representation. Then, the Lagrangian can be constructed by introducing auxiliary superfields as

$$\mathcal{L}_{\text{linear}} = \int d^4\theta \left(\Phi_i^\dagger \Phi_i e^{2V} - cV \right) + \left(\int d^2\theta \, \Phi_0 \Phi_i^2 + \text{ h.c. } \right), \tag{2.1}$$

where $V(x, \theta, \bar{\theta})$ is an auxiliary vector superfield and $\Phi_0(x, \theta, \bar{\theta})$ is an auxiliary chiral superfield belonging to an $SO(N)$ singlet. Here, summation over the index i is implied. The constant c is positive and real, and the term cV is the so-called Fayet-Iliopoulos term. The last two terms constitute the superpotential. This model has four-dimensional $\mathcal{N} = 1$ supersymmetry, global $SO(N)$ symmetry, and $U(1)$ gauge symmetry:

$$\Phi_i(x, \theta, \bar{\theta}) \to e^{i\Lambda(x, \theta, \bar{\theta})} \Phi_i(x, \theta, \bar{\theta}), \qquad \Phi_0(x, \theta, \bar{\theta}) \to e^{-2i\Lambda(x, \theta, \bar{\theta})} \Phi_0(x, \theta, \bar{\theta}),$$

$$e^{2V(x,\theta,\bar{\theta})} \rightarrow e^{2V(x,\theta,\bar{\theta})-i\Lambda(x,\theta,\bar{\theta})+i\Lambda(x,\theta,\bar{\theta})^{\dagger}}. \tag{2.2}$$

Here $\Lambda(x,\theta,\bar{\theta})$ is an arbitrary chiral superfield. The partition function of this model can be written as

$$Z \;=\; \int \mathcal{D}\Phi_i \mathcal{D}\Phi_i^{\dagger} \mathcal{D}\Phi_0 \mathcal{D}\Phi_0^{\dagger} \mathcal{D}V \, \exp\left(i\int d^4x\, \mathcal{L}_{\text{linear}}\right). \tag{2.3}$$

The integration over $V(x,\theta,\bar{\theta})$ gives a D-term constraint, and the Kähler potential becomes nonlinear:

$$\mathcal{L}' = \int d^4\theta \; c\log(\Phi_i^{\dagger}\Phi_i). \tag{2.4}$$

This is the Kähler potential of the Fubini-Study metric of $\mathbf{C}P^{N-1}$ in the homogeneous coordinates Φ_i. (We can show that integration over V is equivalent to the elimination of V by its classical equation of motion [13].) On the other hand, integration over $\Phi_0(x,\theta,\bar{\theta})$ gives the F-term constraint

$$\Phi_i(x,\theta,\bar{\theta})^2 = 0, \tag{2.5}$$

which is holomorphic. $\mathbf{C}P^{N-1}$ with the constraint $\Phi_i^2 = 0$ (on homogeneous coordinates Φ_i) is just the (complex) quadric surface $Q^{N-2}(\mathbf{C})$ [18]. We thus obtain the supersymmetric nonlinear sigma model on $Q^{N-2}(\mathbf{C})$ by integration over the auxiliary superfields V and Φ_0 [12, 13]:

$$\mathcal{L}_{\text{nonlinear}} \;=\; \int d^4\theta\, c\,\log\left\{1 + \varphi_a^{\dagger}\varphi_a + \frac{1}{4}\left(\varphi_a^{\dagger}\right)^2\left(\varphi_a\right)^2\right\}, \tag{2.6}$$

where the fields $\varphi_a(x,\theta,\bar{\theta})$ ($a = 1,\cdots,N-2$) are *nonlinear* dynamical chiral superfields. Here, a solution of the F-term constraint (2.5) on $\Phi_i = (\varphi_a,\alpha,\beta)$ is given by

$$\alpha - i\beta = -\frac{(\varphi_a)^2}{\alpha + i\beta} = -\frac{(\varphi_a)^2}{\sqrt{2}}, \tag{2.7}$$

where we have chosen the specific gauge $\alpha + i\beta = \sqrt{2}$ by using the gauge degrees of freedom, represented by Eq. (2.2).[1] The nonlinear Lagrangian (2.6) coincides with that constructed using the supersymmetric nonlinear realization methods [19, 15, 16]. The target manifold parametrized by (scalar components of) φ_a is the quadric surface and is isomorphic to a Hermitian symmetric space:

$$Q^{N-2}(\mathbf{C}) \;\simeq\; \frac{SO(N)}{SO(N-2)\times U(1)}. \tag{2.8}$$

Although the φ_a transform nonlinearly under $SO(N)$, they transform linearly under the isotropy group $SO(N-2)\times U(1)$, namely as an $SO(N-2)$ vector with an appropriate $U(1)$ charge.

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[1] If we choose another gauge, the Kähler potential in that gauge and Eq. (2.6) can be transformed into each other by the Kähler transformation $K(\varphi,\varphi^{\dagger}) \rightarrow K(\varphi,\varphi^{\dagger}) + f(\varphi) + f^{\dagger}(\varphi^{\dagger})$. The metrics in the two gauges coincide.

Let us construct the two-dimensional model using the notation and reduction rules summarized in Appendix B. The superfields and their component fields are

$$\Phi_i(x,\theta,\overline{\theta}): \ (A_i(x),\psi_i(x),F_i(x)), \qquad \Phi_0(x,\theta,\overline{\theta}): \ (A_0(x),\psi_0(x),F_0(x)),$$

$$V(x,\theta,\overline{\theta}): \ (M(x),N(x),\lambda(x),V_m(x),D(x)), \tag{2.9}$$

where the $\Phi_i(x,\theta,\overline{\theta})$ are dynamical chiral superfields, and $\Phi_0(x,\theta,\overline{\theta})$ and $V(x,\theta,\overline{\theta})$ are auxiliary chiral and vector superfields, respectively. Instead of choosing the gauge (2.7) we have assumed the *Wess-Zumino gauge* for V, suitable for the non-perturbative study. The index $m\,(=0,1)$ labels the two-dimensional space-time coordinates (x^0,x^1). The components fields $N(x)$ and $M(x)$ in V are real scalar fields in two dimensions that originate from components of the gauge field $V_\mu(x)$ in four dimensions, $V^2(x)$ and $V^3(x)$, respectively:

$$N(x) \equiv V^2(x) \ = \ -V_2(x) \,, \quad M(x) \equiv V^3(x) \ = \ -V_3(x) \,.$$

Furthermore, we redefine the gauginos λ, λ^c and complex scalar fields A_0, A_0^* by

$$\lambda \ \to \ \sqrt{2}i\lambda \,, \quad \lambda^c \ \to \ -\sqrt{2}i\lambda^c \,, \tag{2.10a}$$

$$A_0 \ \to \ \frac{1}{2}A_0 \,, \quad A_0^* \ \to \ \frac{1}{2}A_0^* \,, \tag{2.10b}$$

where λ^c is the charge conjugate of λ: $\lambda^c = -\gamma^0\overline{\lambda}^T$. We thus obtain the auxiliary field formulation of the two-dimensional Q^N model in component fields, given by

$$\begin{aligned}
\mathcal{L} \ = \ & F_i^*F_i + \partial_m A_i^* \partial^m A_i + i\overline{\psi}_i\gamma^m\partial_m\psi_i \\
& + V_m\big[iA_i^*\partial^m A_i - i\partial^m A_i^* \cdot A_i + \overline{\psi}_i\gamma^m\psi_i\big] + M\big(\overline{\psi}_i\psi_i\big) - N\big(\overline{\psi}_i i\gamma_3\psi_i\big) \\
& + A_i\big(\overline{\lambda}\psi_i^c + \overline{\psi}_i\lambda^c\big) + A_i^*\big(\overline{\lambda^c}\psi_i + \overline{\psi_i^c}\lambda\big) \\
& + (D + V_mV^m - M^2 - N^2)A_i^*A_i - \frac{1}{2}cD \\
& + \big\{F_0A_i^2 + F_0^*A_i^{*2}\big\} + \big\{F_iA_iA_0 + F_i^*A_i^*A_0^*\big\} \\
& - A_i\big(\overline{\psi_0^c}\psi_i + \overline{\psi_i^c}\psi_0\big) - A_i^*\big(\overline{\psi}_0\psi_i^c + \overline{\psi}_i\psi_0^c\big) - \frac{1}{2}A_0\overline{\psi_i^c}\psi_i - \frac{1}{2}A_0^*\overline{\psi}_i\psi_i^c \,, \tag{2.11}
\end{aligned}$$

where ψ_i^c is the charge conjugate of ψ_i.

We can eliminate the auxiliary fields for supersymmetry, $F_i(x)$ and $F_i^*(x)$, in the dynamical chiral superfields Φ_i by substituting their equations of motion,

$$F_i(x) \ = \ -A_i^*(x)A_0^*(x)\,, \qquad F_i^*(x) \ = \ -A_i(x)A_0(x)\,, \tag{2.12}$$

back into the Lagrangian (2.11):

$$
\begin{aligned}
\mathcal{L} &= \partial_m A_i^* \partial^m A_i + i\overline{\psi}_i \gamma^m \partial_m \psi_i \\
&\quad + V_m \big[iA_i^* \partial^m A_i - i\partial^m A_i^* \cdot A_i + \overline{\psi}_i \gamma^m \psi_i \big] + M\big(\overline{\psi}_i \psi_i\big) - N\big(\overline{\psi}_i i\gamma_3 \psi_i\big) \\
&\quad + A_i\big(\overline{\lambda}\psi_i^c + \overline{\psi}_i \lambda^c\big) + A_i^*\big(\overline{\lambda^c}\psi_i + \overline{\psi_i^c}\lambda\big) \\
&\quad + (D + V_m V^m - M^2 - N^2) A_i^* A_i - \frac{1}{2}cD \\
&\quad + F_0 A_i^2 + F_0^* A_i^{*2} - A_0^* A_0 A_i^* A_i \\
&\quad - A_i\big(\overline{\psi_0^c}\psi_i + \overline{\psi_i^c}\psi_0\big) - A_i^*\big(\overline{\psi}_0 \psi_i^c + \overline{\psi}_i \psi_0^c\big) - \frac{1}{2}A_0 \overline{\psi_i^c}\psi_i - \frac{1}{2}A_0^* \overline{\psi}_i \psi_i^c .
\end{aligned}
\tag{2.13}
$$

Although we start from this Lagrangian (2.13) in the following sections, we discuss the symmetries of (2.11) in the next subsection.

Before discussing symmetries, we eliminate the remaining auxiliary fields. If we eliminate all auxiliary fields using their equations of motion, we obtain the nonlinear Lagrangian

$$
\mathcal{L} = (D_m A_i)^* (D^m A_i) + i\overline{\psi}_i \gamma^m D_m \psi_i + \frac{1}{2c} \left[(\overline{\psi}_i \psi_i)^2 + (\overline{\psi}_i i\gamma_3 \psi_i)^2 + (\overline{\psi}_i \psi_i^c)(\overline{\psi_i^c}\psi_i) \right] ,
\tag{2.14a}
$$

with the constraints

$$
A_i^* A_i = \frac{c}{2}, \qquad A_i^* \psi_i = A_i \overline{\psi}_i = 0, \qquad A_i \psi_i = A_i^* \overline{\psi}_i = 0, \qquad A_i^2 = A_i^{*2} = 0.
\tag{2.14b}
$$

The first (and the second) equations are the same as those of the (supersymmetric) $\mathbf{C}P^N$ model, and the last two are the same as those of the $\mathcal{N} = 1$ supersymmetric $O(N)$ model with "zero radius".[2] In Eq. (2.14a), D_m is the covariant derivative defined by

$$
D_m A_i = (\partial_m - iV_m) A_i , \qquad D_m \psi_i = (\partial_m - iV_m)\psi_i ,
\tag{2.15a}
$$

$$
V_m = \frac{1}{c}\left(iA_i^* \overleftrightarrow{\partial}_m A_i + \overline{\psi}_i \gamma_m \psi_i \right).
\tag{2.15b}
$$

The nonlinear Lagrangian (2.14a) can be obtained from Eq. (2.6) by eliminating auxiliary fields for supersymmetry in $\varphi_a(x, \theta, \overline{\theta})$ (and by dimensional reduction).

2.2 Symmetries

In this subsection we consider symmetries of the two-dimensional Lagrangian (2.11), before eliminating the auxiliary fields F_i and F_i^*. The Lagrangian (2.11) has three types of $U(1)$ symmetries, as described below.

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[2] Since the $A_i(x)$ are complex scalar fields, $A_i^2 = 0$ does not represent a point, but a conifold [16].

1. The local (gauged) $U(1)$ symmetry:

   This is a local phase transformation on superfields under which the Grassmannian coordinates θ are invariant. It is given by

   $$\Phi_i(x,\theta,\overline{\theta}) \;\rightarrow\; e^{i\alpha(x,\theta,\overline{\theta})}\Phi_i(x,\theta,\overline{\theta})\,, \quad \Phi_0(x,\theta,\overline{\theta}) \;\rightarrow\; e^{-2i\alpha(x,\theta,\overline{\theta})}\Phi_0(x,\theta,\overline{\theta})\,, \tag{2.16}$$

   where $\alpha(x,\theta,\overline{\theta})$ is an arbitrary chiral superfield gauge parameter.

2. The global $U(1)$ symmetry:

   This is the global phase transformation on the Grassmannian parameters θ, namely the R symmetry,

   $$\Phi_i(x,\theta,\overline{\theta}) \;\rightarrow\; \Phi_i(x,e^{i\alpha}\theta,e^{-i\alpha}\overline{\theta})\,, \quad \Phi_0(x,\theta,\overline{\theta}) \;\rightarrow\; e^{2i\alpha}\Phi_0(x,e^{i\alpha}\theta,e^{-i\alpha}\overline{\theta})\,, \tag{2.17a}$$

   $$\lambda(x) \;\rightarrow\; e^{i\alpha}\lambda(x)\,. \tag{2.17b}$$

   The origin of this symmetry is the R symmetry in four dimensions.

3. The global chiral $U(1)$ symmetry:

   This $U(1)$ symmetry is another R symmetry whose origin is the rotation in the (x^2,x^3)-plane in four dimensions. In two dimensions, this becomes a chiral symmetry, given by

   $$\Phi_i(x,\theta,\overline{\theta}) \;\rightarrow\; \Phi_i(x,e^{i\gamma_3\alpha}\theta,\overline{\theta}e^{i\gamma_3\alpha})\,, \quad \Phi_0(x,\theta,\overline{\theta}) \;\rightarrow\; \Phi_0(x,e^{i\gamma_3\alpha}\theta,\overline{\theta}e^{i\gamma_3\alpha})\,, \tag{2.18a}$$

   $$\lambda(x) \;\rightarrow\; e^{i\gamma_3\alpha}\lambda(x)\,, \quad M(x)-i\gamma_3 N(x) \;\rightarrow\; e^{-2i\gamma_3\alpha}\big(M(x)-i\gamma_3 N(x)\big)\,. \tag{2.18b}$$

We list the charges of the superfields Φ_i, Φ_0 and V, their component fields, and the Grassmannian variables θ under transformations of these symmetries in Table 2.

symmetries	Φ_i	A_i	ψ_i	F_i	Φ_0	A_0	ψ_0	F_0	V	V_m	$M-i\gamma_3 N$	λ	D	θ
local $U(1)$	1	1	1	1	-2	-2	-2	-2	0	0	0	0	0	0
global $U(1)$	0	0	-1	-2	2	2	1	0	0	0	0	1	0	1
global chiral $U(1)$	0	0	1	0	0	0	1	0	0	0	-2	1	0	1
global + local $U(1)$	1	1	0	-1	0	0	-1	-2	0	0	0	1	0	1

Table 2: $U(1)$ symmetries and their charges.

The last line gives the mixed $U(1)$ symmetry of the global $U(1)$ and the local $U(1)$, which we consider below.

3 Effective Potential and Vacua

In this section, we calculate the effective potential by integrating out all of the dynamical fields $A_i(x)$ and $\psi_i(x)$. From the variations of the effective potential with respect to the vacuum expectation values, we obtain gap equations. By solving them we find two kinds of stable vacua.

3.1 Effective potential and vacuum conditions

We start from the Lagrangian (2.13). We would like to find non-perturbative vacua by analyzing the leading order of the $1/N$ expansion. At the leading order in this expansion, we can neglect quantum fluctuations of auxiliary fields. We set the vacuum expectation values of auxiliary fields as

$$A_0(x) = \phi_0, \qquad A_0^*(x) = \phi_0^*, \qquad F_0(x) = F_c, \qquad F_0^*(x) = F_c^*,$$
$$\psi_0(x) = \overline{\psi}_0(x) = \lambda(x) = \overline{\lambda}(x) = 0,$$
$$M(x) = M_c, \qquad N(x) = N_c, \qquad D(x) = D_c, \qquad V_m(x) = 0. \tag{3.1}$$

Moreover, we decompose the dynamical scalar fields $A_i(x)$ into sums of the classical constant fields ϕ_i and the fluctuating quantum fields $A_i'(x)$ around them, with constraints

$$\int d^2x\, A_i'(x) = 0. \tag{3.2}$$

Since the vacuum expectation values of the dynamical fermionic fields $\psi_i(x)$ are all zero, we express these fluctuating fields also by $\psi_i(x)$.

Let us calculate the effective potential. We substitute the constant fields of the auxiliary fields (3.1) into Eq. (2.13). By integrating out the fluctuating dynamical fields $A_i'(x)$ and $\psi_i(x)$ in the partition function (2.3),

$$Z = \int \mathcal{D}\Phi_0 \mathcal{D}\Phi_0^\dagger \mathcal{D}V \, \exp\left(iS_{\text{eff}}\right), \tag{3.3}$$

we can calculate the effective action S_{eff}, given by

$$S_{\text{eff}} = \frac{iN}{2}\text{Tr}\log\det\left[D_c^{-1}\right] - \frac{iN}{2}\text{Tr}\log\det\left[S_c^{-1}\right] + \int d^2x \mathcal{L}_0. \tag{3.4}$$

Here we have defined

$$D_c^{-1} = \begin{pmatrix} \partial^2 + \phi_0^*\phi_0 - D_c + M_c^2 + N_c^2 & -2F_c^* \\ -2F_c & \partial^2 + \phi_0^*\phi_0 - D_c + M_c^2 + N_c^2 \end{pmatrix}, \tag{3.5a}$$

$$S_c^{-1} = \begin{pmatrix} i\gamma^m\partial_m + M_c \cdot \mathbf{1} - i\gamma_3 N_c & -\phi_0^* \cdot \mathbf{1} \\ -\phi_0 \cdot \mathbf{1} & i\gamma^m\partial_m + M_c \cdot \mathbf{1} + i\gamma_3 N_c \end{pmatrix}, \tag{3.5b}$$

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$$\mathcal{L}_0 = F_c\phi_i^2 + F_c^*\phi_i^{*2} - \phi_0^*\phi_0\phi_i^*\phi_i + (D_c - M_c^2 - N_c^2)\phi_i^*\phi_i - \frac{N}{g^2}D_c . \tag{3.5c}$$

We also have set the Fayet-Iliopoulous constant c as

$$c = \frac{2N}{g^2} , \tag{3.6}$$

by using the coupling constant g and the numbers of dynamical fields A_i and ψ_i, N. In this definition, all terms in the Lagrangian (2.13) become of order N. The effective potential V_{eff} can be calculated from the definition

$$S_{\text{eff}}\big|_{\text{constant fields}} = -V_{\text{eff}} \int d^2x \tag{3.7}$$

to give

$$\begin{aligned}
V_{\text{eff}} =\ & \frac{N}{2} \int \frac{d^2k}{(2\pi)^2i} \log\left[(-k^2 + X^2 + Y^2 - D_c)^2 - 4F_c^*F_c\right] \\
& - \frac{N}{2} \int \frac{d^2k}{(2\pi)^2i} \log\left[(-k^2 + X^2 + Y^2)^2 - 4X^2Y^2\right] \\
& - F_c\phi_i^2 - F_c^*\phi_i^{*2} + (X^2 + Y^2 - D_c)\phi_i^*\phi_i + \frac{N}{g^2}D_c .
\end{aligned} \tag{3.8}$$

Here we have defined X^2 and Y^2 by

$$Y^2 \equiv M_c^2 + N_c^2 , \quad X^2 \equiv \phi_0^*\phi_0 , \tag{3.9}$$

which play the roles of order parameters of the two kinds of vacua, as seen in the following sections.

We can find non-perturbative vacua as the minimum points of the effective potential (3.8). By variations about all of the constant fields ϕ_i, ϕ_i^*, X, Y, F_c, F_c^* and D_c, we obtain the conditions for vacua, which are called the *gap equations*:

$$0 = \frac{N}{g^2} - \phi_i^*\phi_i - N \int \frac{d^2k}{(2\pi)^2i} \frac{-k^2 + X^2 + Y^2 - D_c}{(-k^2 + X^2 + Y^2 - D_c)^2 - 4F_c^*F_c} , \tag{3.10a}$$

$$0 = -\phi_i^2 - 2N \int \frac{d^2k}{(2\pi)^2i} \frac{F_c^*}{(-k^2 + X^2 + Y^2 - D_c)^2 - 4F_c^*F_c} , \tag{3.10b}$$

$$0 = -\phi_i^{*2} - 2N \int \frac{d^2k}{(2\pi)^2i} \frac{F_c}{(-k^2 + X^2 + Y^2 - D_c)^2 - 4F_c^*F_c} , \tag{3.10c}$$

$$0 = \phi_i\{4F_c^*F_c - (X^2 + Y^2 - D_c)^2\} , \tag{3.10d}$$

$$0 = \phi_i^*\{4F_c^*F_c - (X^2 + Y^2 - D_c)^2\} , \tag{3.10e}$$

$$0 = 2X\left\{\frac{N}{g^2} - N \int \frac{d^2k}{(2\pi)^2i} \frac{-k^2 + X^2 - Y^2}{(-k^2 + X^2 + Y^2)^2 - 4X^2Y^2}\right\} , \tag{3.10f}$$

$$0 = 2Y\left\{\frac{N}{g^2} - N \int \frac{d^2k}{(2\pi)^2i} \frac{-k^2 - X^2 + Y^2}{(-k^2 + X^2 + Y^2)^2 - 4X^2Y^2}\right\} . \tag{3.10g}$$

We can find non-perturbatively stable vacua from these equations.

3.2 Supersymmetric vacua

Since we would like to find the supersymmetric vacua, we assume the conditions

$$F_c = F_c^* = D_c = 0 . \tag{3.11}$$

If we can find stable supersymmetric vacua, we need not search for non-supersymmetric (supersymmetry broken) vacua, since non-supersymmetric vacua are unstable if supersymmetric vacua exist. Only when we cannot find any stable vacua preserving supersymmetry under these conditions should we search for non-supersymmetric stable vacua. Substitution of Eq. (3.11) into the gap equations, Eqs. (3.10a)-(3.10g), gives

$$\phi_i[X^2 + Y^2] = 0 , \quad \phi_i^*[X^2 + Y^2] = 0 , \quad \phi_i^2 = \phi_i^{*2} = 0 , \tag{3.12a}$$

$$\frac{N}{g^2} = \phi_i^* \phi_i + N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + X^2 + Y^2} , \tag{3.12b}$$

$$0 = 2X \left\{ \frac{N}{g^2} - N \int \frac{d^2 k}{(2\pi)^2 i} \frac{-k^2 + X^2 - Y^2}{(-k^2 + X^2 + Y^2)^2 - 4X^2 Y^2} \right\} , \tag{3.12c}$$

$$0 = 2Y \left\{ \frac{N}{g^2} - N \int \frac{d^2 k}{(2\pi)^2 i} \frac{-k^2 - X^2 + Y^2}{(-k^2 + X^2 + Y^2)^2 - 4X^2 Y^2} \right\} . \tag{3.12d}$$

The last two equations show that at least X or Y must be zero; these equations are inconsistent with $X \neq 0$ and $Y \neq 0$ holding simultaneously. We thus find two kinds of consistent vacua, $X = 0$ and $Y = 0$. We discuss these below.

1. In the $X = 0$ vacuum, Eq. (3.12c) is trivially satisfied, and the other equations become as follows:

$$(3.12a) \; : \; \phi_i Y^2 = \phi_i^* Y^2 = \phi_i^2 = \phi_i^{*2} = 0 , \tag{3.13}$$

$$(3.12b) \; : \; \frac{N}{g^2} = \phi_i^* \phi_i + N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + Y^2} , \tag{3.14}$$

$$(3.12d) \; : \; 0 = 2Y \left\{ \frac{N}{g^2} - N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + Y^2} \right\} . \tag{3.15}$$

If $Y^2 = 0$, Eq. (3.14) would contain an infrared divergence. Since this divergence is singular, we cannot renormalize it. We thus conclude that $Y^2 \neq 0$. Under this condition, we obtain the final form of the gap equations,

$$\phi_i = \phi_i^* = 0 , \quad \frac{N}{g^2} = N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + Y^2} . \tag{3.16}$$

The second equation is the same as the gap equation of the bosonic $O(N)$ model for zero vacuum expectation values. The value of the effective potential under these conditions is zero:

$$V_{\text{eff}} = 0 . \tag{3.17}$$

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This vacuum has some interesting features. First, it is supersymmetric. Second, the vacuum expectation values ϕ_i of the dynamical scalar fields $A_i(x)$, belonging to an $SO(N)$ vector, are all zero. In perturbation theories, their values are nonzero and $SO(N)$ symmetry is broken. Contrastingly in non-perturbative vacua, we obtain zero vacuum expectation values of $A_i(x)$ and find that $SO(N)$ symmetry is restored. Third, all dynamical fields acquire masses $m = Y$ in order to avoid the infrared divergence. In particular, Dirac fermions $\psi_i(x)$ acquire *Dirac* mass terms. We call this vacuum the "**Schwinger phase**", since the gauge field becomes massive as a result of the Schwinger mechanism, as shown in the next section [20]. This vacuum is the same as that of the $\mathcal{N} = 2$ supersymmetric $\mathbf{C}P^{N-1}$ model.

2. In the $Y = 0$ vacuum, Eq. (3.12d) is trivially satisfied, and the other conditions become

$$(3.12\text{a}) \ : \ \phi_i X^2 \ = \ \phi_i^* X^2 \ = \ \phi_i^2 \ = \ \phi_i^{*2} \ = \ 0 \,, \tag{3.18}$$

$$(3.12\text{b}) \ : \ \frac{N}{g^2} \ = \ \phi_i^* \phi_i + N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + X^2} \,, \tag{3.19}$$

$$(3.12\text{c}) \ : \ 0 \ = \ 2X \left\{ \frac{N}{g^2} - N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + X^2} \right\} \,. \tag{3.20}$$

If $X^2 = 0$, Eq. (3.19) would contain an infrared divergence. Hence, in order to avoid this divergence, we should have $X^2 \neq 0$. We thus obtain the final form of the gap equations,

$$\phi_i \ = \ \phi_i^* \ = \ 0 \,, \quad \frac{N}{g^2} \ = \ N \int \frac{d^2 k}{(2\pi)^2 i} \frac{1}{-k^2 + X^2} \,, \tag{3.21}$$

which are the same as Eq. (3.16) if we replace X^2 by Y^2. The value of the effective potential is again zero:

$$V_{\text{eff}} \ = \ 0 \,. \tag{3.22}$$

This vacuum has also some interesting features. It is supersymmetric and $SO(N)$ symmetric, and there are mass gaps about all fields. In this vacuum, Dirac spinors $\psi_i(x)$ obtain *Majorana* mass terms, in contrast to the Schwinger phase. We call this vacuum the "**Higgs phase**", since a gauge boson acquires mass through the Higgs mechanism. This vacuum had until this time not been seen in two-dimensional nonlinear sigma models.

We thus have found two stable vacua, the Schwinger phase, in which $Y \neq 0$ and $X = 0$, and the Higgs phase, in which $X \neq 0$ and $Y = 0$. The final form of the gap equations, Eqs. (3.16) and (3.21), is the same in the two phases if we replace X by Y and vice versa. The Schwinger phase and the Higgs

phase in this model are similar to the Coulomb and Higgs branches in $\mathcal{N} = 2$ supersymmetric QCD in four dimensions, where scalar components of vector-multiplets and hyper-multiplets acquire vacuum expectation values, respectively. In the following two sections, we calculate two-point functions and β functions in these two vacua.

4   Schwinger Phase

In this section we investigate the Schwinger phase, which is well known as the non-perturbative vacuum of the $\mathcal{N} = 2$ supersymmetric $\mathbf{C}P^{N-1}$ model [8, 9]. In this phase, components of the dynamical superfields, belonging to $SO(N)$ vectors, and components of the auxiliary superfields acquire the non-zero masses $m = |Y|$ and $m = |2Y|$, respectively. We find that all the auxiliary superfields become dynamical as bound states of the original dynamical fields. We also calculate propagators and the β function, and find that this phase is asymptotically free.

4.1   Two-point functions

In this phase, $Y^2 = M_c^2 + N_c^2$ is the only non-zero vacuum expectation value. By using the chiral symmetry Eq. (2.18), this vacuum expectation value can be rotated to M_c. Then vacuum expectation values are given by

$$\phi_0 \;=\; \phi_0^* \;=\; F_c \;=\; F_c^* \;=\; 0 \,, \tag{4.1a}$$

$$N_c \;=\; D_c \;=\; \langle \lambda \rangle \;=\; \langle \psi_0 \rangle \;=\; \langle V_m \rangle \;=\; 0 \,, \tag{4.1b}$$

$$M_c \;=\; -Y \;\neq\; 0 \,. \tag{4.1c}$$

We define M' and N' as the quantum fluctuations of M and N around the above vacuum expectation values. For quantum fluctuations of the remaining fields, the same letters are used, since their vacuum expectation values are all zero. By the vacuum expectation values (4.1), the chiral $U(1)$ symmetry is spontaneously broken, as seen in Table 2. Then N' is a Nambu-Goldstone boson of this breaking, and is massless. (As shown below, this massless boson disappears from the physical spectrum, because it is absorbed by a gauge boson.) Under Eq. (4.1), the Lagrangian (2.13) becomes

$$
\begin{aligned}
\mathcal{L} \;=\; & -A_i^* [\partial^2 + Y^2] A_i + \frac{1}{2} \left(\overline{\psi}_i \,,\, \overline{\psi}_i^c \right) \begin{pmatrix} i\gamma^m \partial_m - Y & 0 \\ 0 & i\gamma^m \partial_m - Y \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_i^c \end{pmatrix} \\
& + V_m \left(i A_i^* \partial^m A_i - \partial^m A_i^* \cdot A_i + \overline{\psi}_i \gamma^m \psi_i \right) + V_m V^m A_i^* A_i \\
& + A_i \left(\overline{\lambda} \psi_i^c + \overline{\psi}_i \lambda^c \right) + A_i^* \left(\overline{\lambda^c} \psi_i + \overline{\psi_i^c} \lambda \right) - A_i \left(\overline{\psi_0^c} \psi_i + \overline{\psi_i^c} \psi_0 \right) - A_i^* \left(\overline{\psi}_0 \psi_i^c + \overline{\psi}_i \psi_0^c \right)
\end{aligned}
$$

$$-\left(-2YM' + M'^2 + N'^2\right)A_i^* A_i + M'\left(\overline{\psi}_i \psi_i\right) - N'\left(\overline{\psi}_i i\gamma_3 \psi_i\right)$$
$$- A_0^* A_0 A_i^* A_i - \frac{1}{2} A_0 \overline{\psi}_i^c \psi_i - \frac{1}{2} A_0^* \overline{\psi}_i \psi_i^c$$
$$+ DA_i^* A_i - \frac{N}{g^2} D + F_0^* A_i^2 + F_0 A_i^{*2} \ . \tag{4.2}$$

From this equation, we find that the original dynamical spinors ψ_i acquire the Dirac mass terms $Y\overline{\psi}_i \psi_i$. The auxiliary spinor ψ_0 also acquires the Dirac mass term, as seen in Table 3, below.

We now expand the effective action to calculate two-point functions. We define two-point functions as coefficients of quadratic terms in an expansion of the effective action by

$$S_{\text{eff}} \ = \ \int \frac{d^2 p}{(2\pi)^2} \sum_{i,j} \widetilde{\mathcal{F}}_i(-p) \Pi_{\mathcal{F}_i \mathcal{G}_j}(p) \widetilde{\mathcal{G}}_j(p) + \cdots \ , \tag{4.3}$$

where $\widetilde{\mathcal{F}}_i(p)$ and $\widetilde{\mathcal{G}}_i(p)$ are arbitrary fields in the momentum representation, and the coefficients $\Pi_{\mathcal{F}_i \mathcal{G}_j}(p)$ are their two-point functions. We thus obtain all of the two-point functions in this phase as listed in Tables 3 and 4. In these tables, for simplicity, we have omitted multiplication by the factor $R(p^2)$ defined by

$$R(p^2) \ = \ \frac{N}{2\pi} \int_0^1 dx \, \frac{1}{Y^2 - x(1-x)p^2} \ . \tag{4.4}$$

In Table 4, the Levi-Civita tensor ϵ^{mn} is defined as

$$\epsilon_{01} \ = \ -\epsilon^{01} \ = \ 1 \ , \quad \epsilon^{mn} \ = \ -\epsilon^{nm} \ , \tag{4.5a}$$

$$\gamma_m \gamma_n \ = \ \eta_{mn} + \epsilon_{mn} \gamma_3 \ , \quad \epsilon_{mn} \epsilon_{kl} \ = \ -\eta_{mk} \eta_{nl} + \eta_{ml} \eta_{nk} \ . \tag{4.5b}$$

In the diagramatics these two-point functions correspond to Feynman diagrams with external lines of auxiliary fields and loops of N dynamical fields A_i or ψ_i, which are listed in Figure 1. We find that the auxiliary fields become dynamical as bound states of the original dynamical fields.

Since the fields D, M', N' and V_m are not diagonal in two-point functions, as seen in Table 4, we should diagonalize them in order to define propagators. If we redefine D and N' by

$$D'(p) \ = \ D(p) - 2YM'(p) \ , \tag{4.6a}$$

$$N''(p) \ = \ N'(p) + 2iY\frac{\epsilon_{mk} p^k}{p^2} V^m(p) \ , \tag{4.6b}$$

we can obtain the diagonal two-point functions with respect to the redefined fields D' and N'', as listed in Table 5.

From Tables 3 and 5, we immediately find that all the fields, except for F_0, D' and N'', acquire masses $m = |2Y|$. N'' is a massless Nambu-Goldstone field for *chiral* $U(1)$ symmetry breaking, and F_0

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$\langle\,\mathcal{F}\,\mathcal{G}\,\rangle$	A_0	A_0^*	ψ_0	ψ_0^c	F_0	F_0^*
A_0^*	$\frac{1}{8}(p^2-4Y^2)$	0	0	0	0	0
A_0	0	$\frac{1}{8}(p^2-4Y^2)$	0	0	0	0
$\overline{\psi}_0$	0	0	$\frac{1}{2}(\not{p}+2Y)$	0	0	0
$\overline{\psi_0^c}$	0	0	0	$\frac{1}{2}(\not{p}+2Y)$	0	0
F_0^*	0	0	0	0	$\frac{1}{2}$	0
F_0^*	0	0	0	0	0	$\frac{1}{2}$

Table 3: Two-point functions of component fields in the chiral superfield Φ_0.

$\mathcal{F}$ and $\mathcal{G}$ denote arbitrary fields. Multiplication by the coefficient $R(p^2)$ defined by Eq. (4.4) is omitted in all components.

$\langle\,\mathcal{F}\,\mathcal{G}\,\rangle$	λ	λ^c	D	M'	N'	V_n
$\overline{\lambda}$	$\frac{1}{2}(\not{p}+2Y)$	0	0	0	0	0
$\overline{\lambda^c}$	0	$\frac{1}{2}(\not{p}+2Y)$	0	0	0	0
D	0	0	$\frac{1}{4}$	$\frac{1}{2}Y$	0	0
M'	0	0	$\frac{1}{2}Y$	$\frac{1}{4}p^2$	0	0
N'	0	0	0	0	$\frac{1}{4}p^2$	$\frac{i}{2}Y\epsilon_{nk}p^k$
V_m	0	0	0	0	$-\frac{i}{2}Y\epsilon_{mk}p^k$	$-\frac{1}{4}(\eta_{mn}p^2-p_mp_n)$

Table 4: Two-point functions of component fields in the vector superfield V.

Note that multiplication by the coefficient $R(p^2)$ defined by Eq. (4.4) is omitted in all components.

$\langle\,\mathcal{F}\,\mathcal{G}\,\rangle$	λ	λ^c	D'	M'	N''	V_n
$\overline{\lambda}$	$\frac{1}{2}(\not{p}+2Y)$	0	0	0	0	0
$\overline{\lambda^c}$	0	$\frac{1}{2}(\not{p}+2Y)$	0	0	0	0
D'	0	0	$\frac{1}{4}$	0	0	0
M'	0	0	0	$\frac{1}{4}(p^2-4Y^2)$	0	0
N''	0	0	0	0	$\frac{1}{4}p^2$	0
V_m	0	0	0	0	0	$-\frac{1}{4}(p^2-4Y^2)\{\eta_{mn}-\frac{p_mp_n}{p^2}\}$

Table 5: Diagonal two-point functions of component fields in the vector superfield V.

Note again that multiplication by the coefficient $R(p^2)$ defined by Eq. (4.4) is omitted in all components.

and D' remain auxiliary fields. We interpret these phenomena as follows. In order to keep the mass relation for $\mathcal{N} = 2$ supersymmetry, the auxiliary field D is mixed with M', and M' becomes massive with mass $|2Y|$. Also, the massless pseudo-scalar boson N' is mixed with a massless gauge boson V_m. As a result, the gauge boson acquires a mass of $|2Y|$. This phenomenon is known as the Schwinger mechanism [20]. Therefore there appear no massless bosons, in agreement with Coleman's theorem. The auxiliary fields, F_0 and D, do not have physical massless poles. Since the Nambu-Goldstone boson N' is absorbed into the gauge boson V_m, N' also has no physical massless pole.

Let us discuss the high-energy behavior. In the high-energy limit $p^2 \to \infty$, all the two-point functions of the auxiliary fields are suppressed, because $R(p^2 \to \infty) \to 0$. This is consistent with the behavior of the auxiliary fields: The auxiliary fields propagate in the low-energy region as bound states, but they do not propagate and disappear in the high-energy region.

Now, let us normalize all the fields properly. In the low-energy limit $p^2 \to 0$, we would like to obtain the normalized two-point functions. For example, we normalize $A_0(x)$ by

$$S_{\text{eff}} \;=\; \int \frac{d^2p}{(2\pi)^2} \widetilde{A}(-p) \Pi_{A_0}(p) \widetilde{A}(p) + \cdots \;=\; \int \frac{d^2p}{(2\pi)^2} \widetilde{A}'(-p) \big\{ Z_A \Pi_{A_0}(p) \big\} \widetilde{A}'(p) + \cdots \;, \tag{4.7}$$

$$Z_A \Pi_{A_0}(p) \;\xrightarrow[p^2 \to 0]{}\; p^2 - 4Y^2 \;, \tag{4.8}$$

$$\frac{1}{8} Z_A R(p^2) \;\xrightarrow[p^2 \to 0]{}\; 1 \;, \quad Z_A \;=\; \frac{16\pi Y^2}{N} \;, \tag{4.9}$$

which fixes the renormalization constant Z_A. We calculate the propagators for normalized fields in the next subsection.

4.2 Propagators and the β function

We now calculate propagators from Tables 3 and 5. In order to define the gauge field propagators, we introduce a covariant gauge fixing term with a gauge parameter α as

$$\langle V_m V_n \rangle \;\equiv\; -(p^2 - 4Y^2) \left\{ \eta_{mn} - (1 - \alpha^{-1}) \frac{p_m p_n}{p^2} \right\} \;. \tag{4.10}$$

We obtain the normalized propagators as

$$D_{A_0}(p) \;=\; \frac{1}{p^2 - 4Y^2} \;, \quad S_{\psi_0}(p) \;=\; \frac{1}{\not{p} + 2Y} \;, \quad D_{F_0}(p) \;=\; 1 \;, \tag{4.11a}$$

$$S_\lambda(p) \;=\; \frac{1}{\not{p} + 2Y} \;, \quad D_{D'}(p) \;=\; 1 \;, \quad D_{M'}(p) \;=\; \frac{1}{p^2 - 4Y^2} \;, \quad D_{N''}(p) \;=\; \frac{1}{p^2} \;, \tag{4.11b}$$

$$D_V^{mn}(p) \;=\; -\frac{1}{p^2 - 4Y^2} \left\{ \eta^{mn} - (1 - \alpha) \frac{p^m p^n}{p^2} \right\} \;, \tag{4.11c}$$

where we have expressed fields in question by the indices on D and S.

Before closing this section, we define the β function. In order to do so, we introduce a cutoff Λ and a renormalization point μ as

$$\frac{1}{g^2} \;=\; \int \frac{d^2k}{(2\pi)^2 i} \frac{1}{-k^2 + Y^2} \;=\; \frac{1}{4\pi} \log \frac{\Lambda^2}{Y^2} \,, \tag{4.12a}$$

$$\frac{1}{g_{\mathrm{R}}^2} \;=\; \frac{1}{g^2} - \frac{1}{4\pi} \log \frac{\Lambda^2}{\mu^2} \;=\; \frac{1}{4\pi} \log \frac{\mu^2}{Y^2} \,. \tag{4.12b}$$

Then we can define the β function $\beta(g_{\mathrm{R}})$ by

$$\beta(g_{\mathrm{R}}) \;=\; \frac{\partial}{\partial \log \mu} \, g_{\mathrm{R}} \;=\; -\frac{g_{\mathrm{R}}^3}{4\pi} \;<\; 0 \,. \tag{4.13}$$

From this equation, it can be found that the system of the Schwinger phase is asymptotically free.
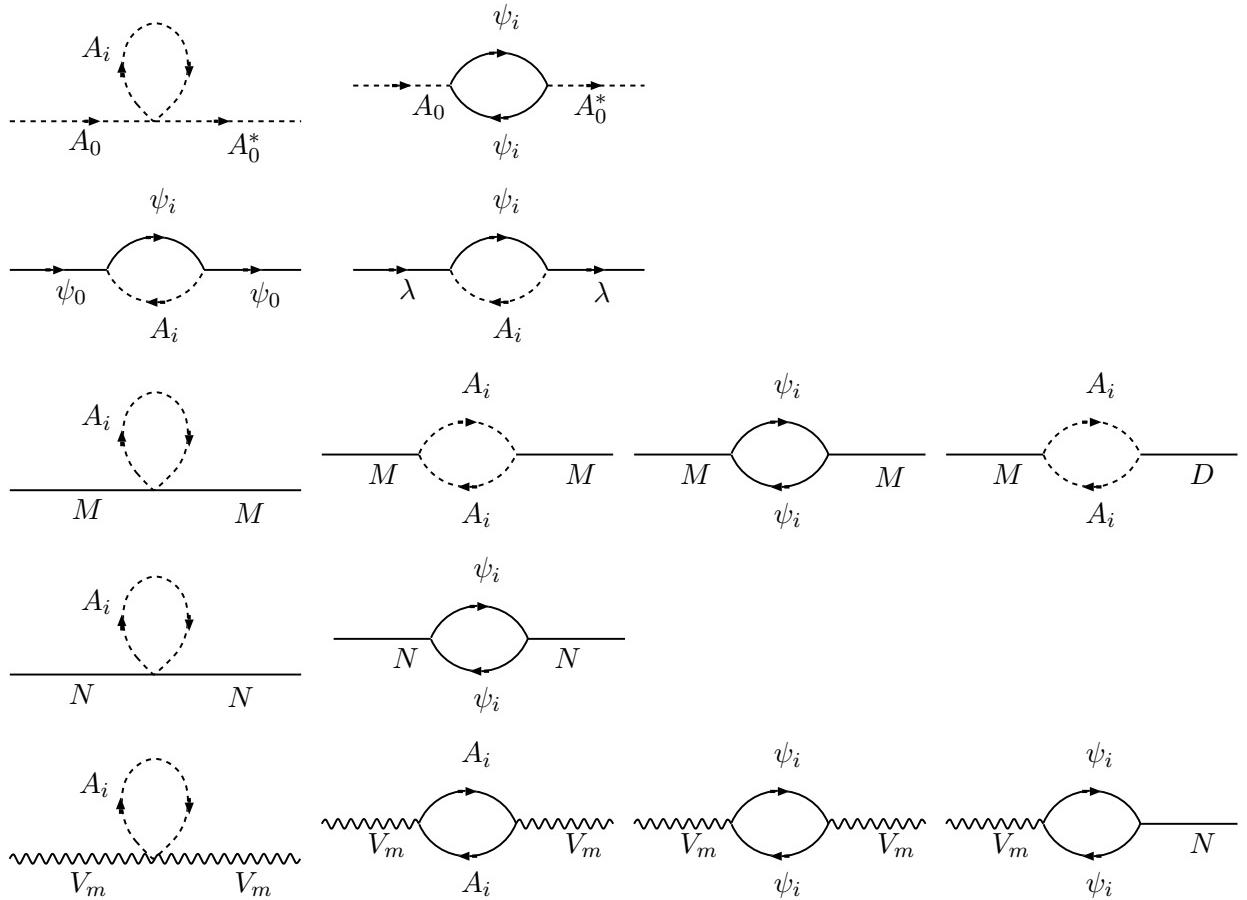

Figure 1: Feynman diagrams for two-point functions of auxiliary fields in the Schwinger phase.

Feynman diagrams with external lines, denoting auxiliary fields, and loops, denoting (integrated) dynamical fields A_i and ψ_i, are listed. All diagrams are order N. We can read how auxiliary fields become bound states of original dynamical fields.

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5   Higgs Phase

We discuss the Higgs phase in this section. This phase has to this time not been known to exist in two-dimensional nonlinear sigma models. It is quite different from the Schwinger phase.

5.1   Two-point functions

The vacuum expectation value $X^2 = \phi_0^* \phi_0$ is non-zero, where $\langle A_0(x) \rangle = \phi_0$. By using a phase rotation, ϕ_0 can be taken to be real: $\phi_0 = X$. We thus decompose the auxiliary fields $A_0(x)$ and $A_0^*(x)$ into sums of X and the quantum fluctuations by

$$A_0(x) \ = \ X + A_R(x) + iA_I(x) \,, \quad A_0^*(x) \ = \ X + A_R(x) - iA_I(x) \,. \tag{5.1}$$

The other fields do not acquire vacuum expectation values. We again define M' and N' as quantum fluctuations of M and N, and for the rest of the fields we use the same letters for the quantum fluctuations as those used for the original fields. By the vacuum expectation value (5.1), the global $U(1)$ symmetry and the local $U(1)$ symmetry are broken down to their linear combination in the last line of Table 2. Since the unbroken symmetry is a global symmetry, the local $U(1)$ symmetry is broken. $A_I(x)$ becomes a (would-be) Nambu-Goldstone boson for the symmetry breaking of the local $U(1)$ symmetry. We can rewrite the Lagrangian as

$$
\begin{aligned}
\mathcal{L} \ = \ & -A_i^* \big[\partial^2 + X^2 \big] A_i + \frac{1}{2} \Big(\overline{\psi}_i \,, \ \overline{\psi}_i^c \Big) \begin{pmatrix} i\gamma^m \partial_m & -X \\ -X & i\gamma^m \partial_m \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_i^c \end{pmatrix} \\
& + V_m \Big(iA_i^* \partial^m A_i - \partial^m A_i^* \cdot A_i + \overline{\psi}_i \gamma^m \psi_i \Big) + V_m V^m A_i^* A_i \\
& + A_i \Big(\overline{\lambda} \psi_i^c + \overline{\psi}_i \lambda^c \Big) + A_i^* \Big(\overline{\lambda^c} \psi_i + \overline{\psi}_i^c \lambda \Big) \\
& - M'^2 A_i^* A_i + M' \Big(\overline{\psi}_i \psi_i \Big) - N'^2 A_i^* A_i - N' \Big(\overline{\psi}_i i\gamma_3 \psi_i \Big) - A_i \Big(\overline{\psi}_0^c \psi_i + \overline{\psi}_i^c \psi_0 \Big) - A_i^* \Big(\overline{\psi}_0 \psi_i^c + \overline{\psi}_i \psi_0^c \Big) \\
& - \Big(2XA_R + A_R^2 + A_I^2 \Big) A_i^* A_i - \Big(A_R + iA_I \Big) \overline{\psi}_i^c \psi_i - \Big(A_R - iA_I \Big) \overline{\psi}_i \psi_i^c \\
& + DA_i^* A_i - \frac{N}{g^2} D + F_0^* A_i^2 + F_0^* A_i^{*2} \,. \tag{5.2}
\end{aligned}
$$

From this equation, we find that the dynamical spinors ψ_i acquire the Majorana mass terms $X\overline{\psi}_i^c \psi_i + X\overline{\psi}_i \psi_i^c$. The auxiliary spinor ψ_0 also acquires the Majorana mass term, as seen in Table 6, below.

Let us calculate the two-point functions in the Higgs phase. The definitions of the two-point functions are the same as those in the Schwinger phase. The coefficients of the two-point functions, $R(p^2)$, and the two-point functions of the gauge fields, $Q_{mn}(p)$, are defined by

$$R(p^2) \ = \ \frac{N}{2\pi} \int_0^1 dx \ \frac{1}{X^2 - x(1-x)p^2} \,, \tag{5.3a}$$

$$Q_{mn}(p) = \left\{ X^2 \eta_{mn} - \frac{1}{4}(\eta_{mn}p^2 - p_m p_n) \right\} R(p^2) = \tilde{Q}_{mn}(p) R(p^2) . \tag{5.3b}$$

We list all the two-point functions in Table 6, where we have again omitted the coefficient $R(p^2)$ for simplicity. To diagonalize the two-point functions in Table 6, we should redefine fields by

$$A'_I(p) = A_I(p) + \frac{2iXp^m}{p^2} V_m(p) , \quad D'(p) = D(p) - 2X A_R(p) , \tag{5.4a}$$

$$\psi'_0(p) = \psi_0(p) + \lambda^c(p) , \quad \lambda'(p) = \lambda(p) - \psi_0^c(p) . \tag{5.4b}$$

The diagonal two-point functions are listed in Tables 7 and 8. (Note again that we have omitted the coefficient $R(p^2)$ in these tables.) Since other fields are already diagonal, they are not given in these tables. The auxiliary fields, F_0 and D, do not have physical massless poles. The field A_I, which is the Nambu-Goldstone boson for local $U(1)$ symmetry breaking, is absorbed into the gauge boson V_m, and therefore it has no physical massless pole.

We now discuss the high-energy behavior in the Higgs phase. In the high-energy limit $p^2 \to \infty$, all two-point functions of the auxiliary fields are suppressed, because $R(p^2 \to \infty) \to 0$, as in the Schwinger phase. We should normalize the auxiliary fields in the low-energy limit. For example, we normalize $A_R(x)$ by

$$S_{\text{eff}} = \int \frac{d^2p}{(2\pi)^2} \widetilde{A}_R(-p) \Pi_{A_R}(p) \widetilde{A}_R(p) + \cdots = \int \frac{d^2p}{(2\pi)^2} \widetilde{A}'_R(-p) \{ Z_A \Pi_{A_R}(p) \} \widetilde{A}'_R(p) + \cdots , \tag{5.5}$$

$$Z_A \Pi_{A_R}(p) \xrightarrow[p^2 \to 0]{} p^2 - 4X^2 , \tag{5.6}$$

$$\frac{1}{4} Z_A R(p^2) \xrightarrow[p^2 \to 0]{} 1 , \quad Z_A = \frac{8\pi X^2}{N} . \tag{5.7}$$

Let us discuss the phenomena exhibited in this phase. In order to keep $\mathcal{N} = 2$ supersymmetry, a massless boson A_R is mixed with D, and then A_R obtains mass $m = |2X|$. The Nambu-Goldstone boson A_I is absorbed into a massless gauge boson V_m to form a massive gauge boson with mass $m = |2X|$, as a result of the Higgs mechanism. Dirac fermions ψ_0 and λ are mixed with each other and acquire masses $m = |2X|$. Since F_0 and D' are independent of p^2 [but depend on $R(p^2)$], they remain auxiliary fields.

To summarize, the original dynamical field have acquired mass $m = |X|$, and the $SO(N)$ symmetry is restored. In addition, all auxiliary fields, except for F_0 and D', have become dynamical as bound states of the original dynamical fields, whose masses are all $m = |2X|$.

$\langle\mathcal{FG}\rangle$	A_R	A_I	ψ_0	ψ_0^c	F_0^*	F_0	D	λ	λ^c	M'	N'	V_n
A_R	$\frac{1}{4}p^2$	0	0	0	0	0	$-\frac{1}{2}X$	0	0	0	0	0
A_I	0	$\frac{1}{4}p^2$	0	0	0	0	0	0	0	0	0	$-\frac{i}{2}p_n X$
$\overline{\psi}_0$	0	0	$\frac{1}{2}\not{p}$	0	0	0	0	0	$-X$	0	0	0
$\overline{\psi}_0^c$	0	0	0	$\frac{1}{2}\not{p}$	0	0	0	$-X$	0	0	0	0
F_0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0
F_0^*	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0
D	$-\frac{1}{2}X$	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
$\overline{\lambda}$	0	0	0	$-X$	0	0	0	$\frac{1}{2}\not{p}$	0	0	0	0
$\overline{\lambda}^c$	0	0	$-X$	0	0	0	0	0	$\frac{1}{2}\not{p}$	0	0	0
M'	0	0	0	0	0	0	0	0	0	$\frac{1}{4}(p^2-4X^2)$	0	0
N'	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}(p^2-4X^2)$	0
V_m	0	$\frac{i}{2}p_m X$	0	0	0	0	0	0	0	0	0	$\tilde{Q}_{mn}(p)$

Table 6: The two-point functions in the Higgs phase.

Multiplication by the coefficient $R(p^2)$ defined by Eq. (5.3a) is omitted in all components.

$\langle\mathcal{F}\mathcal{G}\rangle$	A_R	D'	A'_I	V_n
A_R	$\frac{1}{4}(p^2-4X^2)$	0	0	0
D'	0	$\frac{1}{4}$	0	0
A'_I	0	0	$\frac{1}{4}p^2$	0
V_m	0	0	0	$-\frac{1}{4}(p^2-4X^2)\{\eta_{mn}-\frac{p_m p_n}{p^2}\}$

Table 7: The two-point functions of A_R, D', A_I, V_m.

$\langle\mathcal{F}\mathcal{G}\rangle$	ψ'_0	$\lambda^{c\prime}$	$\mathcal{F}\psi_0^{c\prime}$	λ'
$\overline{\psi}'_0$	$\frac{1}{2}(\not{p}+2X)$	0	0	0
$\overline{\lambda^{c\prime}}$	0	$\frac{1}{2}(\not{p}-2X)$	0	0
$\overline{\psi_0^{c\prime}}$	0	0	$\frac{1}{2}(\not{p}+2X)$	0
$\overline{\lambda}'$	0	0	0	$-\frac{1}{2}(\not{p}-2X)$

Table 8: The two-point functions of ψ'_0 and λ.

5.2 Propagators and the β function

In this subsection we calculate normalized propagators. We first introduce a covariant gauge fixing term with a gauge parameter α in order to construct propagators of gauge fields by

$$\langle V_m V_n\rangle \equiv -(p^2-4X^2)\Big\{\eta_{mn}-(1-\alpha^{-1})\frac{p_m p_n}{p^2}\Big\}\,. \tag{5.8}$$

We can calculate all of the normalized propagators. We have the following:

$$D_{M'}(p) = \frac{1}{p^2-4X^2}\,, \qquad D_{N'}(p) = \frac{1}{p^2-4X^2}\,, \tag{5.9a}$$

$$S_{\psi'_0}(p) = \frac{1}{\not{p}-2X}\,, \qquad S_{\lambda'}(p) = \frac{1}{\not{p}-2X}\,, \tag{5.9b}$$

$$D_{A_R}(p) = \frac{1}{p^2-4X^2}\,, \quad D_{A'_I}(p) = \frac{1}{p^2}\,, \tag{5.9c}$$

$$D_V^{mn}(p) = -\frac{1}{p^2-4X^2}\Big\{\eta^{mn}-(1-\alpha)\frac{p^m p^n}{p^2}\Big\}\,, \tag{5.9d}$$

$$D_{D'}(p) = 1\,, \quad D_{F_0}(p) = 1\,, \tag{5.9e}$$

where we have expressed fields in question by the indices on D and S.

To define the β function, we introduce a cutoff Λ and a renormalization point μ as

$$\frac{1}{g^2} = \int \frac{d^2k}{(2\pi)^2 i}\frac{1}{-k^2+X^2} = \frac{1}{4\pi}\log\frac{\Lambda^2}{X^2}\,, \tag{5.10a}$$

20

$$\frac{1}{g_{\mathrm{R}}^2} \;=\; \frac{1}{g^2} - \frac{1}{4\pi}\log\frac{\Lambda^2}{\mu^2} \;=\; \frac{1}{4\pi}\log\frac{\mu^2}{X^2} \;. \tag{5.10b}$$

We thus define the β function $\beta(g_{\mathrm{R}})$ by

$$\beta(g_{\mathrm{R}}) \;=\; \frac{\partial}{\partial\log\mu}\, g_{\mathrm{R}} \;=\; -\frac{g_{\mathrm{R}}^3}{4\pi} \;<\; 0\;, \tag{5.11}$$

which shows that the system of the Higgs phase is also asymptotically free.
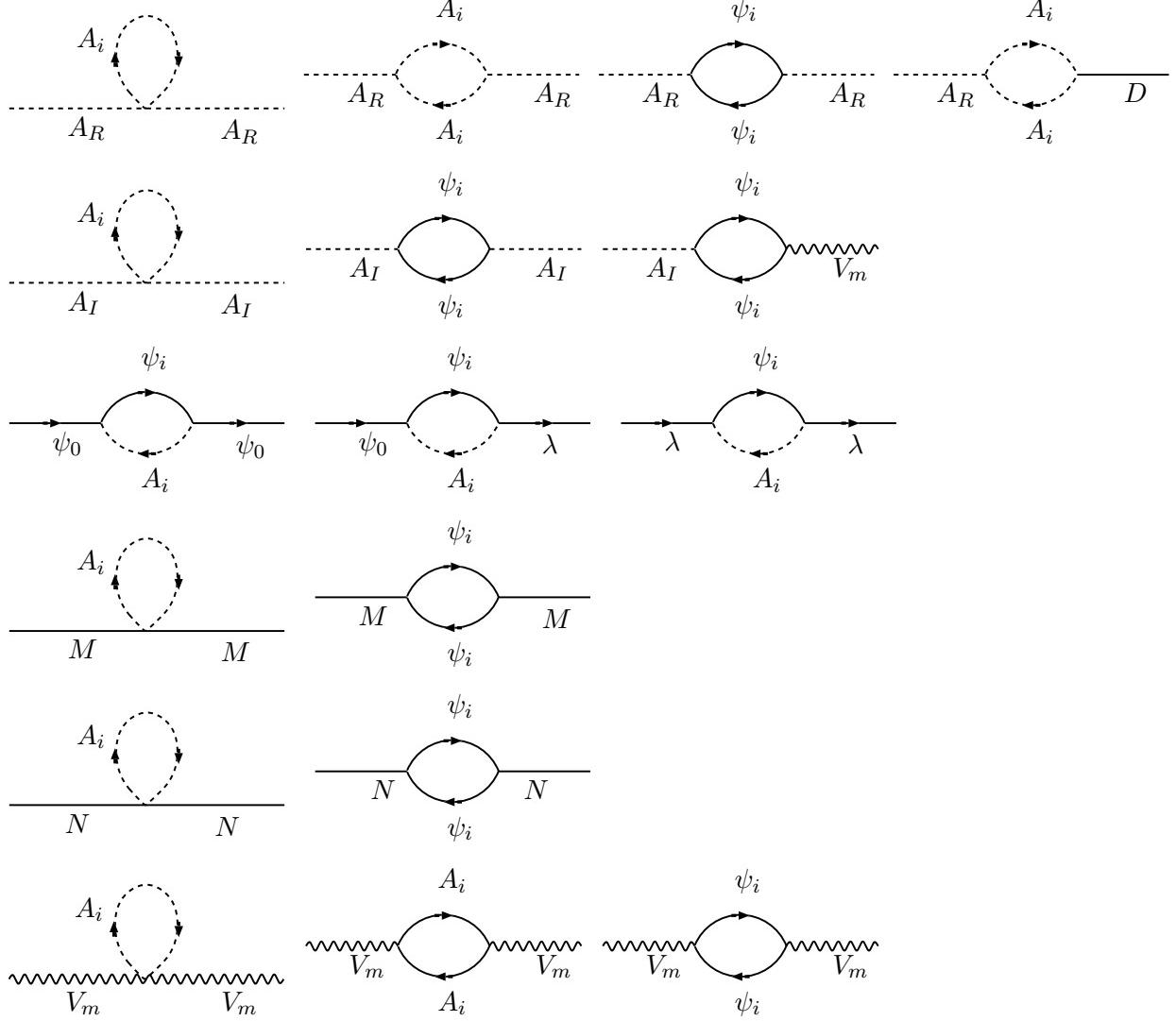
Figure 2: Feynman diagrams for two-point functions of auxiliary fields in the Higgs phase.

Feynman diagrams with external lines, denoting auxiliary fields, and loops, denoting (integrated) dynamical fields A_i and ψ_i, are listed. All diagrams are order N. We can read how auxiliary fields become bound states of original dynamical fields.

6 Conclusion and Discussion

We have studied non-perturbative effects of the two-dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma model on the quadric surface $Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$ (the Q^N model), by using auxiliary field and large-N methods. To formulate the Q^N model by auxiliary field methods, we needed two kinds of auxiliary superfields, a *vector* superfield $V(x, \theta, \overline{\theta})$ and a *chiral* superfield $\Phi_0(x, \theta, \overline{\theta})$. By integrating out the dynamical fields $A_i(x)$ and $\psi_i(x)$ [and $F_i(x)$], we calculated the effective potential. We have found that this model has two kinds of non-perturbatively stable vacua. In these vacua, scalar components of the auxiliary *vector* and *chiral* superfields, namely $M(x)$ [or $N(x)$] and $A_0(x)$, acquire non-zero vacuum expectation values, given by Y and X, respectively. The former is the same vacuum as that of the $\mathcal{N} = 2$ supersymmetric $\mathbf{C}P^{N-1}$ model. We call it the *Schwinger phase*, since a massless gauge boson $V_m(x)$ becomes massive as a result of the Schwinger mechanism. The latter is a new kind of vacuum, which has been seen here for the first time. We call it the *Higgs phase*, since a massless gauge boson becomes massive due to the Higgs mechanism.

In the Schwinger phase, all component fields of the dynamical chiral superfields $\Phi_i(x, \theta, \overline{\theta})$, belonging to $SO(N)$ vectors, acquire masses $m = |Y| = \langle M(x) \rangle$, and the $SO(N)$ symmetry is dynamically restored. In particular, Dirac spinors $\psi_i(x)$ acquire *Dirac* mass terms, which break the global *chiral* $U(1)$ symmetry spontaneously. Then, one of the auxiliary fields, $N(x)$, becomes a massless Nambu-Goldstone boson. In two dimensions, however, the appearance of a Nambu-Goldstone boson is forbidden by Coleman's theorem. The massless gauge field $V_m(x)$ and $N(x)$ are mixed, and the massless pseudo-scalar $N(x)$ is absorbed into the gauge boson as a result of the Schwinger mechanism. In addition, the auxiliary field $M(x)$ becomes massive through mixing with $D(x)$ to preserve supersymmetry. Therefore, all massless bosons disappear from the physical spectrum in agreement with Coleman's theorem. Then, all the component fields of the auxiliary superfields, except for auxiliary fields needed for supersymmetry, acquire masses $m = |2Y|$, and supersymmetry is preserved.

In the Higgs phase, all component fields of the dynamical superfields, belonging to the $SO(N)$ vector representation, obtain masses $m = |X|$, and the $SO(N)$ is again dynamically restored. Here, the complex fermions $\psi_i(x)$ acquire *Majorana* mass terms. By these mass terms, the $U(1)_{\text{global}} \times U(1)_{\text{local}}$ symmetry is broken to their linear combination. In this phase, the imaginary part of the scalar field $A_0(x)$, $A_I(x)$, becomes a (would-be) Nambu-Goldstone boson. This massless boson, however, is absorbed into the gauge boson $V_m(x)$ to form a massive gauge boson, as a result of the Higgs mechanism. The real part of $A_0(x)$, $A_R(x)$, becomes massive through mixing with $D(x)$. Therefore, all massless bosons again disappear from the physical spectrum in agreement with Coleman's theorem.

All component fields of the auxiliary superfields, except for auxiliary fields for supersymemtry, acquire masses $m = |2X|$, and supersymmetry is preserved.

Furthermore we have shown that both phases are asymptotically free by calculating the β functions.

In this paper, we have discussed the leading order of the $1/N$ expansion. It is interesting to consider the next order of this model, in particular whether there exist next-to-leading order corrections of the $1/N$ expansion. Scalar components of auxiliary vector and chiral superfields acquire non-zero vacuum expectation values in the Schwinger and Higgs phases, respectively. They are similar to the Coulomb and Higgs branches of four-dimensional $\mathcal{N} = 2$ supersymmetric QCD in the sense that scalar components of gauge multiplets and hyper-multiplets acquire non-zero vacuum expectation values. Further investigation of the similarities to four-dimensional $\mathcal{N} = 2$ supersymmetric QCD would be interesting.

Let us now discuss possible generalizations of this model. We would like to discuss non-perturbative effects of nonlinear sigma models on other Hermitian symmetric spaces summarized in Table 1 in the Introduction. For example, non-Abelian gauge bosons would be dynamically generated in the Grassmannian model, the $SO(2N)/U(N)$ model, and the $Sp(N)/U(N)$ model. It is also an interesting task to generalize this model to three dimensions. In three dimensions, nonlinear sigma models are perturbatively non-renormalizable, but they are renormalizable in the $1/N$ expansion. Recently there has been progress in the study of $\mathcal{N} = 2$ ($\mathcal{N} = 4$) supersymmetric nonlinear sigma models on $\mathbf{C}P^{N-1}$ (the cotangent bundle over $\mathbf{C}P^{N-1}$) in three dimensions [21]. By dimensional reduction of our model in four dimensions [12] to three dimensions, we would be able to treat other models in three dimensions. We hope that the investigation of these new models in two (or three) dimensions would provide us further understanding of non-perturbative aspects of quantum field theories.

Acknowledgements

A   Notation for $\mathcal{N} = 1$ Supersymmetry in Four Dimensions

Before constructing $\mathcal{N} = 2$ supersymmetry in two dimensions in the next appendix, we here define the notation for the spinors and $\mathcal{N} = 1$ supersymmetry in four dimensions [22]. The space-time metric is $\eta_{\mu\nu} = \text{diag.}(-+++)$. The Majorana spinors ψ^M can be written by the Weyl spinors as

$$\psi^M = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix} , \qquad \overline{\psi^M} = \left(-\psi^\alpha, -\overline{\psi}_{\dot\alpha} \right) . \tag{A.1}$$

The Dirac matrices in four dimensions are

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\overline{\sigma}^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix} , \quad \sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} , \quad (A.2)$$

where σ^i are the Pauli matrices: $\overline{\sigma}^0 = \sigma^0$, $\overline{\sigma}^i = -\sigma^i$. We note the identity

$$\psi\sigma^\mu\overline{\chi} = -\overline{\chi}\,\overline{\sigma}^\mu\psi , \quad \overline{\psi} = \psi^\dagger \tag{A.3}$$

for the Weyl spinors.

We now give the notation for superfields. A chiral superfield, satisfying $\overline{D}_{\dot\alpha}\phi(x,\theta,\overline{\theta}) = 0$, is

$$\phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),$$

$$\phi(x,\theta,\overline{\theta}) = A(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu A(x) + \frac{1}{4}\theta\theta\overline{\theta}\,\overline{\theta}\Box A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\overline{\theta} + \theta\theta F(x) , \quad (A.4)$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$. A vector superfield, satisfying $V(x,\theta,\overline{\theta})^\dagger = V(x,\theta,\overline{\theta})$, is

$$V(x,\theta,\overline{\theta}) = -\theta\sigma^\mu\overline{\theta}V_\mu + i\theta\theta\overline{\theta}\,\overline{\lambda}(x) - i\overline{\theta}\,\overline{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta}\,\overline{\theta}D(x) \tag{A.5}$$

in the Wess-Zumino gauge. To perform the dimensional reduction, we must consider the Fermion bilinear forms

$$\overline{\psi_i^M}\psi_i^M = -\psi_i\psi_i - \overline{\psi}_i\overline{\psi}_i , \tag{A.6a}$$

$$\overline{\psi_i^M}\gamma_5\psi_i^M = i\psi_i\psi_i - i\overline{\psi}_i\overline{\psi}_i , \tag{A.6b}$$

$$\overline{\psi_i^M}\gamma^\mu\psi_i^M = -\psi_i\sigma^\mu\overline{\psi}_i - \overline{\psi}_i\overline{\sigma}^\mu\psi_i = 0 , \tag{A.6c}$$

$$\overline{\psi_i^M}\gamma^\mu\gamma_5\psi_i^M = -i\psi_i\sigma^\mu\overline{\psi}_i + i\overline{\psi}_i\overline{\sigma}^\mu\psi_i = 2i\overline{\psi}_i\overline{\sigma}^\mu\psi_i , \tag{A.6d}$$

$$\overline{\psi_i^M}\gamma^\mu\partial_\mu\psi_i^M = -\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - \overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i = -2\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i , \tag{A.6e}$$

$$\overline{\psi_i^M}\gamma^\mu\gamma_5\partial_\mu\psi_i^M = i\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - i\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i = 0 . \tag{A.6f}$$

Here integration over x is implied for each equation.

B   Dimensional Reduction to Two Dimensions

Before dimensional reduction, we change some notation. First, we change the sign of the space-time metric according to

$$\eta_{\mu\nu} = \text{diag.}(-+++) = -\text{diag.}(+---) = -\widetilde{\eta}_{\mu\nu}. \tag{B.1}$$

Next, we change the Dirac gamma matrices γ^μ from those in Appendix A to ours:

$$\gamma^\mu = \Gamma^\mu , \quad i\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \Gamma_5 , \tag{B.2a}$$

$$\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu} = 2\widetilde{\eta}^{\mu\nu} = \{\Gamma^\mu,\Gamma^\nu\} . \tag{B.2b}$$

We denote the Dirac matrices in four dimensions and in two dimensions by Γ^μ and γ^m, respectively. The latter can be embedded in the former as follows:

$$\Gamma^m = \gamma^m \otimes \sigma_1 = \begin{pmatrix} 0 & \gamma^m \\ \gamma^m & 0 \end{pmatrix} , \quad m = 0,1 , \tag{B.3a}$$

$$\Gamma^2 = i\gamma_3 \otimes \sigma_1 = \begin{pmatrix} 0 & i\gamma_3 \\ i\gamma_3 & 0 \end{pmatrix} , \quad \Gamma^3 = \mathbf{1} \otimes i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -\mathbf{1} & 0 \end{pmatrix} , \tag{B.3b}$$

$$\Gamma_5 = \mathbf{1} \otimes \sigma_3 , \quad C_4 = i\Gamma^1\Gamma^2 = \gamma^0 \otimes \mathbf{1} = \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix} = -C_2 \otimes \mathbf{1} , \tag{B.3c}$$

$$\gamma^0 = \sigma_2 , \quad \gamma^1 = i\sigma_1 , \quad \gamma_3 = \gamma^0\gamma^1 = \sigma_3 , \tag{B.3d}$$

$$C_2 = -\gamma^0 , \tag{B.3e}$$

$$C_2 = -C_2^T = -C_2^* = C_2^\dagger = C_2^{-1} , \tag{B.3f}$$

$$C_2^{-1}\gamma^\mu C_2 = -\gamma^{\mu T} , \quad C_2^{-1}\gamma_3 C_2 = -\gamma_3^T . \tag{B.3g}$$

The four-dimensional Majorana spinor ψ^M can be expressed by the two-component Weyl spinor ψ as $\psi^M = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$. Then the Majorana condition $\psi^M = C_4 \overline{\psi^M}^T$ becomes

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} -C_2\overline{\psi'}^T \\ -C_2\overline{\psi}^T \end{pmatrix} = \begin{pmatrix} \psi \\ -\chi \end{pmatrix} , \qquad \chi = C_2\overline{\psi}^T = \psi^* . \tag{B.4}$$

If the Majorana spinor ψ^M in four dimensions does not depend on x^2 and x^3, we can rewrite this spinor with the two-dimensional Dirac spinor ψ. The two-component Weyl spinors ψ and χ in four

dimensions become Dirac spinors in two dimensions when we apply dimensional reduction from $\{x^\mu\}$ to $\{x^m\}$ ($\mu = 0, 1, 2, 3$; $m = 0, 1$). Fermion bilinear forms become

$$\overline{\psi_i^M}\psi_j^M \;=\; -\overline{\psi_i^c}\psi_j - \overline{\psi}_i\psi_j^c \;, \tag{B.5a}$$

$$\overline{\psi_i^M}\Gamma_5\psi_j^M \;=\; -\overline{\psi_i^c}\psi_j + \overline{\psi}_i\psi_j^c \;, \tag{B.5b}$$

$$\overline{\psi_i^M}\Gamma^m\psi_i^M \;=\; \overline{\psi_i^M}\Gamma^2\psi_i^M \;=\; \overline{\psi_i^M}\Gamma^3\psi_i^M \;=\; 0 \;, \tag{B.5c}$$

$$\overline{\psi_i^M}\Gamma^m\Gamma_5\psi_i^M \;=\; 2\overline{\psi}_i\gamma^m\psi_i \;, \tag{B.5d}$$

$$\overline{\psi_i^M}\Gamma^2\Gamma_5\psi_i^M \;=\; 2i\overline{\psi}_i\gamma_3\psi_i \;, \tag{B.5e}$$

$$\overline{\psi_i^M}\Gamma^3\Gamma_5\psi_i^M \;=\; -2\overline{\psi}_i\psi_i \;, \tag{B.5f}$$

$$\overline{\psi_i^M}\Gamma^m\partial_m\psi_i^M \;=\; 2\overline{\psi}_i\gamma^m\partial_m\psi_i \;, \tag{B.5g}$$

$$\overline{\psi_i^M}\Gamma^2\partial_2\psi_i^M \;=\; 2i\overline{\psi}_i\gamma_3\partial_2\psi_i \;, \tag{B.5h}$$

$$\overline{\psi_i^M}\Gamma^3\partial_3\psi_i^M \;=\; -2\overline{\psi}_i\partial_3\psi_i \;, \tag{B.5i}$$

$$\overline{\psi_i^M}\Gamma^\mu\Gamma_5\partial_\mu\psi_i^M \;=\; 0 \;, \tag{B.5j}$$

where the integral over x for each equation is implied.

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